1.3. CONTINUITY FOR FUNCTION OF SEVERAL VARIABLE

1.3.1. Introduction

A function f(x, y) is continuous at a point (a, b) *if*,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) \quad ....(1)$$

Then the limit of f as(x, y) tends to (a, b) = the value of f at (a, b). If the function is continuous at every point in the domain, a function is said to be continuous in the domain.

Equation (1) can also be written as:

$$\lim_{(h,k)\to(0,0)} f(a+h,b+k)=f(a,b)$$

If f is not continuous at (a, b) then it is discontinuous at (a, b).

Definition: If $\lim_{(x,y)\to(a,b)} = f(a,b)$ then a function f of two variables is **continuous** at (a, b). If f is continuous at every point (a, b) in D then we can say that f is continuous on D.

Result: If f(x, y) and g(x, y) are continuous at (a, b) then $f \pm g$, $f \cdot g$ and f/g are continuous at (a, b).

Test for Continuity at a Point (a, b)

Step I: f(a, b) should be well defined.

Step II: lim f(x, y) as (x, y) → (a, b) should exists (must be unique and same along any path).

Step III: The limit of f = value of f at (a, b).

1.3.1.1. Uniform Continuity

Let f : A→R, where A ⊂ R. Then f is uniformly continuous on A if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that;

$|x - y| < \delta$ and $x, y \in A$ implies that $|f(x) - f(y)| < \varepsilon$.

In this definition δ is based on ε, not on x, y. At every point of A, a uniformly continuous function on A is continuous but its converse is false.

A detail explanation of this point is given as if a function f is continuous on A, then we have $\varepsilon > 0$ and $c \in A$, there exists $\delta(\varepsilon, c) > 0$ such that

$|x - c| < \delta(\varepsilon, c)$ and $x \in A$ implies that $|f(x) - f(c)| < \varepsilon$.

For some $\varepsilon_0 > 0$ we have

$$\inf_{c \in A} \delta(\varepsilon_0, c) = 0$$

The function is continuous on A but not uniformly continuous because if we select $\delta(\varepsilon_0, c) > 0$ in the definition of continuity, then no $\delta_0(\varepsilon_0) > 0$ is based only on ε_0 which applied simultaneously for every $c \in A$

exists $\epsilon_0 > 0$ and sequences (x_n), (y_n) in A such that $\lim_{n\to\infty}|x_n - y_n| = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n \in N$.

Proof: If f is not uniformly continuous, then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points $x, y \in A$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$. Selecting $x_n, y_n \in A$ to be any such points for $\delta = 1/n$, we obtained the required sequences.

On the other hand, if the sequential condition holds, then for every $\delta > 0$ there exists $n \in N$ such that $|x_n - y_n| < \delta$

and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. It follows that the uniform continuity condition in Definition can not hold for any $\delta > 0$ if $\varepsilon = \varepsilon_0$, so f is not uniformly continuous.

For example:

1) Express f: [0, 1] → R by $f(x) = x^2$. Then f is uniformly continuous on [0, 1]. Verify this, for all x, $y \in [0, 1]$ as:

$$\left|x^2 - y^2\right| = |x + y||x - y| \leq 2|x - y|,$$

In the definition of uniform continuity take $\delta = \varepsilon/2$. Also, $f(x) = x^2$ is uniformly continuous on any bounded set.

2) The function $f(x) = x^2$ is continuous but not uniformly continuous on R. Since we proved it before that f is continuous on R (it is a polynomial). To show that f is not uniformly continuous, suppose

$$x_n = n,\ y_n = n + \frac{1}{n}. \Rightarrow \lim_{n\to\infty} |x_n - y_n| = \lim_{n\to\infty} \frac{1}{n} = 0,$$

But,

$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} \geq 2 \text{ for every } n \in N.$$

From above statement, it obeys that f is not uniformly continuous on R. Now the problem is that in order to show the continuity of f at c, we have $\varepsilon > 0$ we need to make $\delta(\varepsilon, c)$ smaller as c gets larger, and $\delta(\varepsilon, c) \to 0$ as $c \to \infty$.

3) The function f : (0, 1] → R defined by $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous on (0, 1] because it is a rational function whose denominator x is non-zero in (0, 1]. To prove that f is not uniformly continuous, we define $x_n, y_n \in (0, 1)$ for $n \in N$ by,

$$x_n = \frac{1}{n}, \qquad y_n = \frac{1}{n+1}.$$

Then, $|x_n - y_n| \to 0$ as $n \to \infty$, but

$$|f(x_n) - f(y_n)| = (n + 1) - n = 1 \quad \text{for every } n \in N.$$

From above statement, it obeys that f is not uniformly continuous on (0, 1). The problem here is that we have $\varepsilon > 0$, we need to make $\delta(\varepsilon, c)$ smaller as c gets closer to 0, and $\delta(\varepsilon, c) \to 0$ as $c \to 0^+$.

Note: However, if bounded continuous functions oscillate arbitrarily quickly even bounded continuous functions can fail to be uniformly continuous.